I J R B A T, Issue (XI) Vol (I) Jan 2023: 329-332

A Double-Blind Peer Reviewed & Refereed Journal

OPEN ACCESS

e-ISSN 2347 – 517X

Original Article

INTERNATIONAL JOURNAL OF RESEARCHES IN BIOSCIENCES, AGRICULTURE AND TECHNOLOGY

INVERSION THEOREM ASSOCIATED WITH LINEAR CANONICAL-MELLIN TRANSFORM

Vidya Sharma[1] and Nitin Dhange[2]

[1]Department of Mathematics, Smt. Narsamma Arts, Commerce & Science College Amravati, (M.S.) India.

[2]Department of Mathematics, Bhawabhuti Mahavidyalaya, Amgaon, (M.S.) India.

Communicated :10.12.2022	Revision: 20.01.2023 & 24.01.2023 Accepted: 26.01.2023	Published: 30.01.2023

ABSTRACT:

The Linear canonical-Mellin transform is a mixed type of integral transform in which function is both linear canonical and Mellin transformable. Extension of some transformations to generalized functions have been done time to time and their properties have been studied by various mathematicians. However, there is much scope in extending double transformations to a certain class of generalized functions. In this paper, Linear Canonical-Mellin transform is extended in the distributional generalized sense and inversion theorem for Linear Canonical-Mellin transform is proved which can be used to retrieve original function to be transformed.

Keywords: - Inversion theorem, Linear canonical transform, Mellin transform, Testing function space.

INTRODUCTION :

Linear canonical-Mellin transform is a mixed type of integral transform with composition of linear canonical and Mellin transform in which linear canonical transform (LCT) is an integral transform with generalized kernel and Mellin transform is a basic integral transform. The LCT is a four-parameter family of integral transform defined by [1]:

$$L_A[\emptyset](u) = \Phi(u) = \begin{cases} \int_{-\infty}^{\infty} \emptyset(t)\, K_A(u,t)dt, & b \neq 0 \\ \sqrt{d}e^{j\frac{cd}{2}u^2}\emptyset(du), & b = 0 \end{cases}$$

where the LCT kernel $K_A(u,t)$ is given by the operator $K_A(u,t) = \frac{1}{\sqrt{j2\pi b}} e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]}$ and parameters a, b, c, d are real numbers satisfying $ad - bc = 1$. On condition that the parameters satisfy $b = 0$, the LCT is essentially a scaling and chirp multiplication operations. Without loss of generality, we therefore focus mainly on the LCT in the case of $b \neq 0$. In that case, the inverse LCT is

$$\emptyset(t) = \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \Phi(u)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} du.$$

It is easy to verify that the LCT with parameters $(a, b, c, d) = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ reduces to the fractional Fourier transform (FRFT), which, in the specific case $\theta = \frac{\pi}{2}$, becomes the Fourier transform and Parseval relation for LCT is given by [2]:

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}dt = \int_{-\infty}^{\infty} F(u)\,\overline{G(u)}du$$

The Mellin transform is defined as [3]:

$$M[f;s] \equiv F(s) = \int_0^{\infty} f(x)\, x^{s-1} dx$$

and its Parseval relation is given by

$$\int_0^{\infty} f(x)\,\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(1-s)\,\overline{G(s)}ds$$

Chii-Huei Yu [4] provided a new technique to determine some definite integrals using Parseval's theorem and this technique can be applied to solve another definite integral problems. Soo-Chang Pei [5,6] derived many important properties of discrete fractional

http://doi.org/10.29369/ijrbat.2023.010.1.0057

Fourier transform and discussed some applications, such as the filter design and pattern recognition Zemanian [7] studied several integral transforms in the distributional generalized sense. Sharma et. al. [8, 9] had generalized many integral transforms to the distribution of compact support and provided some operational properties of two-dimensional fractional Mellin transform, two dimensional fractional Fourier-Mellin transform. The aim of this paper is to prove inversion formula of Linear Canonical-Mellin transform and Parseval's theorem.

Linear Canonical-Mellin Transform (LCMT)

Definition: The conventional Linear Canonical-Mellin transform is defined as follows:

$$L_A M\{f(t,x)\} = F^A M(u,s) = \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, K(t,x,u,s)dtdx$$

where $K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2-\left(\frac{2}{b}\right)tu+\frac{d}{b}u^2\right]}x^{s-1}$, $b \neq 0, s > 0$.

The Testing Function Space $E(R^n)$

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$, if for each compact set $K \subset S_a$, $I \subset S_b$, where $S_a = \{t: t \in R^n, |t| \leq a, a > 0\}$ and $S_b = \{x: x \in R^n, |x| \leq b, b > 0\}$, $K, I \in R^n$,

$$\gamma_{E,l,q} = \sup_{\substack{t\in K\\ x\in I}} \left|D_t^l D_x^q \varphi(t,x)\right| < \infty, \quad l,q = 0,1,2,---$$

Thus $E(R^n)$ will denote the space of all $\varphi \in E(R^n)$ with support contained in S_a and S_b. Moreover, we say that f is a linear canonical-Mellin transformable if it is a member of E^*, the dual space of E.

Distributional Generalized Linear Canonical-Mellin Transform

The distributional Linear Canonical-Mellin transform of $f(t,x) \in E^*(R^n)$ is defined by

$$L_A M\{f(t,x)\} = F^A M(u,s) = \langle f(t,x), K(t,x,u,s)\rangle \quad (2.1)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ and

$$K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2-\left(\frac{2}{b}\right)tu+\frac{d}{b}u^2\right]}x^{s-1},\ b \neq 0, s > 0.$$

The right-hand side of (1) is meaningful because $K(t,x,u,s) \in E$ and $f(t,x) \in E^*$.

Inversion Formula:

If Linear Canonical-Mellin transform of $f(t,x)$ is given by

$$L_A M\{f(t,x)\} = F^A M(u,s) = \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)}x^{s-1}dtdx$$

Then its inverse $f(t,x)$ is given by

$$f(t,x) = \frac{1}{2\pi}\sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^A M(u,s)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)}x^{-s}duds$$

Proof: - By definition, we have

$$L_A M\{f(t,x)\} = F^A M(u,s) = \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)}x^{s-1}dtdx$$

$$\Rightarrow \sqrt{2j\pi b}F^A M(u,s)e^{\frac{-jd}{2b}u^2} = \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, e^{\frac{ja}{2b}t^2} e^{-j\frac{u}{b}t}x^{s-1}dtdx$$

$$\Rightarrow C_1(u,s) = \int_{-\infty}^{\infty}\int_0^{\infty} g(t,x)\, e^{-j\frac{u}{b}t}x^{s-1}dtdx$$

(3.1)

where,

$$C_1(u,s) = \sqrt{2j\pi b}F^A M(u,s)e^{\frac{-jd}{2b}u^2}$$

and

$$g(t,x) = f(t,x)e^{\frac{ja}{2b}t^2}$$

From (3.1),

$$C_1(u,s) = FMT\{g(t,x)\}\left(\frac{u}{b},s\right)$$

Where, $FMT\{g(t,x)\}\left(\frac{u}{b},s\right)$ is Fourier-Mellin transform of $g(t,x)$ with argument $\frac{u}{b} = w$ and s.

$$\Rightarrow \frac{du}{b} = dw$$

$$\therefore C_1(u,s) = FMT\{g(t,x)\}(w,s)$$

i.e.,

$$C_1(u,s) = \int_{-\infty}^{\infty}\int_0^{\infty} g(t,x)\, e^{-jwt}x^{s-1}dtdx$$

http://doi.org/10.29369/ijrbat.2023.010.1.0057

By using inversion formula for Fourier-Mellin transform, we get

$$g(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_1(u,s)\, e^{jwt} x^{-s} dwds$$

$$\Rightarrow g(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{2j\pi b} F^A M(u,s) e^{\frac{-jd}{2b}u^2}\, e^{jwt} x^{-s} dwds$$

$$= \frac{\sqrt{2j\pi b}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{\frac{-jd}{2b}u^2}\, e^{j\frac{u}{b}t} x^{-s} \frac{du}{b} ds$$

$$= \frac{1}{2\pi} \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{\frac{-jd}{2b}u^2}\, e^{j\frac{u}{b}t} x^{-s} duds$$

$$\Rightarrow f(t,x) e^{\frac{ja}{2b}t^2} = \frac{1}{2\pi} \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{\frac{-jd}{2b}u^2 + j\frac{u}{b}t} x^{-s} duds$$

$$\Rightarrow f(t,x) = \frac{1}{2\pi} \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds$$

Parseval's Theorem for Linear Canonical-Mellin Transform:

Theorem: If $L_A M\{f(t,x)\} = F^A M(u,s)$ and $L_A M\{g(t,x)\} = G^A M(u,s)$, then

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, \overline{g(t,x)} dtdx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u, 1-s)\, \overline{G^A M(u,s)} duds$$

Proof: - We have, by definition

$$L_A M\{f(t,x)\} = F^A M(u,s)$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dtdx$$

By using inversion formula, we get

$$f(t,x) = \frac{1}{2\pi} \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds$$

$$\Rightarrow \overline{f(t,x)} = \frac{1}{2\pi} \sqrt{\frac{-j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{F^A M(u,s)}\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds$$

$$= \frac{1}{2\pi} \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{F^A M(u,s)}\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds$$

And

$$\overline{g(t,x)} = \frac{1}{2\pi} \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{G^A M(u,s)}\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds$$

Consider,

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, \overline{g(t,x)} dtdx$$

$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x) \left\{ \frac{1}{2\pi} \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{G^A M(u,s)}\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} duds \right\} dtdx$$

By changing the order of integration, we get

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, \overline{g(t,x)} dtdx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{G^A M(u,s)} \left\{ \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} dtdx \right\} duds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{G^A M(u,s)} \left\{ \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{(1-s)-1} dtdx \right\} duds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u, 1-s)\, \overline{G^A M(u,s)} duds$$

CONCLUSION :

In this paper, we proved inversion formula associated with Linear Canonical-Mellin transform. Subsequently, Parseval's theorem is proved using inversion formula. The Parseval's theorem is helpful in signal processing, studying behaviours of random processes and relating functions from one domain to another. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems.

REFERENCES:

Zhi-Chao Zhang, "Linear canonical transform's differentiation properties and their application in solving generalized differential equations", Optik-International Journal for Light and Electron Optics 188, pp. 287-293, 2019.

Adrian Stern, "Uncertainty principles in linear canonical transform domains and some

http://doi.org/10.29369/ijrbat.2023.010.1.0057

of their implications in optics", J. Opt. Soc. Am. A, Vol. 25, No. 3, 2008.

Debnath L., Bhatta D., "Integral Transforms and Their Applications", second edition, New York, 2007.

Chii-Huei Yu, "Application of Parseval's Theorem on Evaluating Some Definite Integrals", Turkish Journal of Analysis and Number Theory, Vol. 2, No. 1, pp. 1-5, 2014.

Soo-Chang Pei, "Closed-Form Discrete Fractional and Affine Fourier Transforms", IEEE transactions on signal processing, vol. 48, no. 5, 2000.

Soo-Chang Pei, Min-Hung Yeh, and Tzyy-Liang Luo, "Fractional Fourier Series Expansion for Finite Signals and Dual Extension to Discrete-Time Fractional Fourier Transform", IEEE transactions on signal processing, vol. 47, no. 10, 1999.

Zemanian A. H., "Generalized Integral Transform", Inter Science Publishers, New York, 1968.

Sharma V.D., Khapre S. A., "Applications on Generalized Two-Dimensional Fractional Cosine Transform", International Journal of Engineering and Innovative Technology.

Sharma V. D., Deshmukh P. B., "Operation Transform Formulae for Two Dimensional Fractional Fourier-Mellin Transform", International Journal of Science and Research, pp. 634-637, 2012.

http://doi.org/10.29369/ijrbat.2023.010.1.0057